Solutions to the Renormalization Group Equations for Yukawa Matrices as an Answer to the Quark and Lepton Mass Problem

Bipin R. Desai and Alexander R. Vaucher

University of California, Riverside, California 92507

February 1, 2008

Abstract

If the scale dependence of a Yukawa matrix is assumed to be determined entirely by the dominant 33-element, then the renormalization group equations can be expressed in terms of two separate equations: a differential equation for the 33-coupling, and, an algebraic equation for the scale-independent 3x3 matrix that is found to have only two non-trivial, hierarchical, solutions with eigenvalues (0,0,1) and (0,1,1). The mass matrices are constructed from these solutions by rotating them first by the experimentally known mixing matrices-the CKM for quarks and charged leptons, and the CKM-analog for the seesaw generated Majorana neutrinos-and then incorporating the appropriate texture zeros. A uniform, hierarchical, description for the mass matrices of quarks and leptons is thus achieved, in terms of the mixing paramters, that give mass eigenvalues consistent with experiments as well as reproduce the input mixing angles. Inverted hierarchy in neutrinos is also discussed. Only a single scale $\left(\approx 10^{13}\ GeV\right)$ for the seesaw neutrinos is involved rather than their mass distribution. No new particles are otherwise invoked.

I. Introduction

The Yukawa matrices which describe the mass values in the quark sector of the standard model show a pronounced hierarchical pattern, as do the mixing angles of the CKM matrix [1, 2],

In the lepton sector the charged lepton masses show a similar pattern of hierarchy. The neutrinos, on the other hand, are massless in the minimal version of the standard model, as no right-handed neutrinos are assumed to exist. The neutrino oscillation data [3,4,5,6,7,8] indicate, however, that neutrinos have masses which are extremely small and have a pattern which can be consistent with hierarchy or, alternatively, inverted hierarchy [9].

The mixing angles, the CKM-analog [10], for leptons for a diagonal charged lepton mass matrix are quite different. While for CKM, all three rotation angles are small, here two of the angles are found to be quite large[3,4,5,6,7,8].

We would like to address all these facts together and find a uniform, underlying explanation by solving the renormalization group equations in MSSM (minimal supersymmetric standard model) [11,12] supplemented by two assumptions which reflect physics beyond the standard model. We assume the well known seesaw mechanism [13] for generating the small (Majorana) neutrino masses in which the standard model left-handed neutrinos couple to large mass right-handed neutrinos. And we take account of the fact that texture zeros may be present in the Yukawa matrices of quarks, charged leptons and neutrinos [1,11,14,15,16,17,18,19].

We begin first in, sections II and III, with discussing the renormalization group equations (RGE) of the Yukawa matrices.The RGEs, to one loop in MSSM, for up-quark (U) and down-quark (D) Yukawa matrices are given as follows [11]

$$\frac{dU}{dt} = \frac{1}{16\pi^2}\left[-\sum_i c_i g_i^2 + 3UU^+ + DD^+ + Tr(3UU^+)\mathbf{1}\right]U \tag{1.1}$$

$$\frac{dD}{dt} = \frac{1}{16\pi^2}\left[-\sum_i c_i' g_i^2 + 3DD^+ + UU^+ + Tr(3DD^+)\mathbf{1}\right]D \tag{1.2}$$

where $t = ln(\mu\ /1Gev)$, μ being the energy variable, c_i and c_i' are known constants and $g_i's$ are the gauge couplings.

In the lepton sector, we consider the charged lepton Yukawa matrix, E, and, for the neutrinos we assume a seesaw mechanism, as mentioned earlier, and consider the (Dirac) neutrino matrix, N, that couples the standard model left-handed neutrinos to the large-mass right-handed seesaw neutrinos given by the mass distribution, M_R.To one loup in MSSM the RGEs are given by [11,12]

$$\frac{dE}{dt} = \frac{1}{16\pi^2}\left[-\sum_i d_i g_i^2 + 3EE^+ + NN^+ + Tr(EE^+)\mathbf{1} + Tr(3DD^+)\mathbf{1}\right]E \tag{1.3}$$

$$\frac{dN}{dt}=\frac{1}{16\pi^2}\left[-\sum_i d_i' g_i^2+3NN^{+}+EE^{+}+Tr(NN^{+})\mathbf{1}+Tr(3UU^{+})\mathbf{1}\right]U \tag{1.4}$$

where $d_i$and d_i' are known constants. The Majorana neutrino mass matrix, κ, is then given by the seesaw formula [12,13]

$$\kappa=N^T M_R^{-1} N \tag{1.5}$$

Renormalization group equations, used as a tool to determine the properties of the quark Yukawa matrices have been considered previously, specfically for the case in which the top coupling is assumed dominanat[1,14,20]. Recently, the RGEs have also been used to reconcile with the neutrino data and to investigate the differences between CKM and the CKM-analog mixing angles [10,12,21]

Let us consider the U matix given by (1.1). Assuming the 33-matrix element to be dominant and equal to the top-quark Yukawa coupling, λ_t,we obtain the following equation ignoring D and the $g_i's$ as they are small compared to λ_t

$$\frac{d\lambda_t}{dt}=\frac{3\lambda_t^3}{8\pi^2} \tag{1.6}$$

The solution is given by,

$$\lambda_t\left(t\right)=\lambda_{0t}\left[1+\frac{3}{4\pi^2}\left(t_0-t\right)\lambda_{0t}^2\right]^{-\frac{1}{2}} \tag{1.7}$$

where we have taken $\lambda_t\left(t_0\right)=\lambda_{0t}$ at a convenient scale parameter t_0.

The question we now wish to explore is this: just as the magnitude of λ_t dominates the U-matrix, are there solutions to (1.1) such that the scale-dependence, $\lambda_t\left(t\right)$ given by (1.7) also describes the scale-dependence of the entire U- matrix ? In other words, does equation (1.1) allow a factorizable solution of the type

$$U=U_0\lambda_t\left(t\right) \tag{1.8}$$

that the t-dependence, as given by $\lambda_t\left(t\right)$, can be factored out, leaving behind a matrix of coefficients U_0 that is independent of the scale, t. Similarly, we wish to explore the possibility that the solutions to (1.2), (1.3) and (1.4) can also be expressed in a factorizable form with the scale-dependence residing entirely in the dominant matrix element of each, given by the b-quark, $\tau-$lepton, and the largest mass (Dirac) neutrino, m_3 (or $m_1$for inverted hierarchy) e.g.

$$D=D_0\lambda_b\left(t\right) \tag{1.9a}$$

$$E = E_0 \lambda_\tau (t) \tag{1.9b}$$

$$N = N_0 \lambda_{3v} (t) \tag{1.9c}$$

Such solutions, indeed, exist as we will discuss below, in which the RGEs split into two equations, one a differential equation for the dominant (scale-dependent) Yukawa coupling and the other an algebraic equation for the (scale-independent) matrix of the coefficients. It is found that the scale-independent matrices (e.g. U_0, D_0 etc.) can be classified in terms of their eigenvalues in simple, diagonal –"primordial"– forms.

In deriving these results we assume U, D, N and κ to be real and symmetric.

There are, actually only two non-trivial solutions for the scale-independent matrices : one we designate as "hierarchical", and the other "semi-hierarchical", of the forms

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad and \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.10}$$

respectively. In particular, U_0 is identified as the former, D_0 as the mixture of the two and similarly for the leptons. A general, real, symmetic, solution can be obtained from this "primordial" matrix by rotating it through arbitrary angles by a unitary matrix. That is, if U_{diag} and D_{diag} are related to the "primordial" matrices then one can write the general matrices as

$$U_0 = V U_{diag} V^\dagger \tag{1.11}$$

$$D_0 = V D_{diag} V^\dagger \tag{1.12}$$

where V is a unitary matrix. A similar situation exits for the leptons.

Thus the Yukawa matrix elements will depend directly on the rotation parameters of V.

In section IV, confining first to the quark sector, what we find most remarkable is that, if we assume the unitary matrix V to be the same as the CKM matrix, we obtain Yukawa matrices U_0 and D_0 that are exactly what one generally expects when expressed in powers of the Cabibbo parameter, λ [1]. Namely,

$$U_0 \approx \begin{bmatrix} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{bmatrix} \tag{1.13}$$

Similarly for D_0 where the hierarchy is found to be less pronounced, just as expected [1].

In the lepton sector, discussed in V, the solutions for the charged leptons exhibit a similar hierarchy through the small angles of the CKM matrix. For the neutrinos, we argue that for the mass matrix, κ, defined in (1.5) the candidate for the appropriate unitary matrix V is the large-angle lepton mixing matrix, the CKM-analog [10].

We thus show that for each of the four systems of particles, the mass matrix can be described in terms of the appropriate mixing parameters through an expression that is very simple and transparent.

Although our model reproduces the expected hierarchical pattern of the mass matrix elements, it is, at this stage, incomplete, because, for example, the eigenvalues of U_0 and D_0 in equation (1.11) and (1.12) are still given by their "primordial" representatios i.e. $(0, 0, 1)$ and $(0, 1, 1)$. Furthermore, the mixing matrix turns out to be a unit matrix, since the same matrix, V, that diagonalizes U_0, also diagonalizes D_0.

All that is remedied, however, once we incorporate texture zeros, as we do in section V, whereby, in U_0 and D_0, constructed through equations (1.11) and (1.12), some of the entries are replaced by zeros.

Confining ourselves, briefly, to the quark sector we note that the (symmetric) Yukawa matrices U_0 and D_0 that are consistent with experiments and allow the maximum number of texture zeros, are found to be of the following type,

$$U_0 = \begin{bmatrix} 0 & 0 & X \\ 0 & X & 0 \\ X & 0 & X \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{bmatrix} \tag{1.14}$$

a total of five texture zeros, where X denotes non-zero entries[16.17]. We assume this to be the pattern of the zeros in our case.

Two things happen as soon as texture zeros are incorporated. First, the eigenvalues values of (the newly textured) U_0 and D_0 will no longer be given simply by the primordial values, but will be proportional to the matrix elements of V that were involved in (1.11) and (1.12). Second, the mixing matrix will not necessarily be a unit matrix since there will now be a mismatch between matrices that diagonalize U_0 and D_0, as is evident from the structures in (1.14)

We need then to define two types of mixing matrices :$V^{(in)}_{CKM}$ and $V^{(out)}_{CKM}$.

The "input" mixing matrix, $V^{(in)}_{CKM}$, represents V which generates Yukawa matrices U_0 and D_0 through equations (1.11) and (1.12). The entries in these matrices are the experimentally known CKM parameters.

After texture zeros are incorporated, according to the structures (1.14), an "output" CKM, $V^{(out)}_{CKM}$ is defined which is a mixing matrix between (the newly textured) U_0 and D_0. That is, if V_u and V_d diagonalize the two (newly textured) matrices U_0 and D_0, respectively, then,

$$V^{(out)}_{CKM} = V_u^{\dagger} V_d \tag{1.15}$$

If $V^{(out)}_{CKM}$ turns out to be the same as the experimentally known CKM matrix

(i.e. if $V_{CKM}^{(out)} = V_{CKM}^{(in)}$) then it is a success of the model, justifying the choice of $V_{CKM}^{(in)}$ as the unitary matrix V in (1.11) and (1.12). That is, indeed, found to be the case when we carry out our calculations.

In other words, what was put in to generate the mass matrices, came out, in a self-consistent manner, when we calculated the mixing angles.

At the same time, we also find that the eigenvalues, or rather, the ratios of the eigenvalues to the 33-elements are consistent with the experimentally determined mass ratios.

For the leptons, discussed in VI and VII, we take the charged lepton structure for E_0 to be the same as D_0. And for the neutrinos we refer to an extensive analysis of the texture in the neutrino mass matrix which points to two possible types of structures that are consistent with experiments[18]. The so called A-type structure given by

$$\kappa_0 = \begin{bmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{bmatrix} \quad or \ \kappa_0 = \begin{bmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{bmatrix} \tag{1.16}$$

gives hierarchical neutrino mass values, and the C-type

$$\kappa_0 = \begin{bmatrix} X & X & X \\ X & 0 & X \\ X & X & 0 \end{bmatrix} \tag{1.17}$$

gives an inverted hierarchy [18,19].

As in the case of quarks, as we discuss in section V, we have $V_{CKM}^{(in)}$ that generates charged leptons Yukawa matrices, E_0,while for κ_0, that role, we argue, will be played by $V_{large}^{(in)}$ which is the large angle mixing matrix obtained from the neutrino oscillation data [3,4,5,6,7,8,10] in the basis where the charged leptons are diagonal. And, $V_{large}^{(out)}$ will be an appropriate product of the matrix that diagonalizes κ_0, and the one which diagonalizes E_0 since the E_0 matrix we obtain from our prescription will not necessarily be diagonal.

In other words, the process is the same as for U_0 and D_0. We first create E_0 and κ_0 by the appropriately designated V. Then we replace some of the entries by zeros as given above and examine the newly textured matrices.

Once again we find that, with the A-type structure, $V_{large}^{(out)}$ is the same as $V_{large}^{(in)}$, except for a small discrepancy in the the angle s_2 for the 1-3 sub matrix. The mass eigenvalues are found to be consistent with experiments.

We discuss the C-type structure for κ_0 separately, in section VIII, since it produces an inverted hierarchy which can not be accomodated within the framwork of the "primordial" matrices we have pursued which are basically hierarchical in nature. Instead we point out that, for this case, the appropriate

basis is provided by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.18}$$

We find that, as a consequence, $V_{large}^{(out)}$ is essentially identical to $V_{large}^{(in)}$ with a maximal atmospheric angle mixing (2-3 submatrix), and $s_2 = 0$. The neutrino mass eigenvalues are also consistent with experiments.

II.The RG Equations

First let us define the following more convenient variable

$$x = \frac{3}{4\pi^2}\left(t_0 - t\right) \tag{2.1}$$

then, (1.6) and (1.7) can be expressed as

$$-\frac{d\lambda_t}{dx} = \frac{1}{2}\lambda_t^3 \tag{2.2}$$

$$\lambda_t\left(x\right) = \lambda_{0t}\left[1 + x\lambda_{0t}^2\right]^{-\frac{1}{2}} \tag{2.3}$$

Secondly, ignoring the gauge terms, which are small, we re-arrange the remaining terms in (1.1) and (1.2), as follows

$$-\frac{dU}{dx} = \frac{1}{4}\left[UU^+ + Tr(UU^+)\mathbf{1}\right]U + \frac{1}{12}DD^+U \tag{2.4}$$

$$-\frac{dD}{dx} = \frac{1}{4}\left[DD^+ + Tr(DD^+)\mathbf{1}\right]D + \frac{1}{12}UU^+D \tag{2.5}$$

where the first term, in the square brackets in each, involves "uncoupled" terms, and the second involves coupling between U and D.

In a similar fashion we write, E and N in the following form

$$-\frac{dE}{dx} = \frac{1}{12}\left[3EE^+ + Tr(EE^+)\mathbf{1}\right]E \;+\frac{1}{12}NN^+E \quad +\frac{1}{4}Tr(DD^+)\mathbf{1}E \tag{2.6}$$

$$-\frac{dN}{dx} = \frac{1}{12}\left[3NN^+ + Tr(NN^+)\mathbf{1}\right]N \;+\frac{1}{12}EE^+N \quad +\frac{1}{4}Tr(UU^+)\mathbf{1}N \tag{2.7}$$

where, besides the uncoupled and coupling terms, we have a term, in each, which contains a trace in the quark sector that basically normalizes (re-scales) the solutions.

III. Solutions to the RGEs and the eigenvalues in the quark sector

The equations involving only the uncoupled terms in both (2.4) and (2.5) look like

$$-\frac{dM}{dx}=\frac{1}{4}\left[MM^{+}+Tr(MM^{+})\mathbf{1}\right]M \tag{3.1}$$

For a solution of the type (1.8) and (1.9a)

$$M=M_0\lambda_m\left(x\right) \tag{3.2}$$

$\lambda_m\left(x\right)$ representing the dominant 33-matrix element, we have two equations

$$-\frac{d\lambda_m\left(x\right)}{dx}=\frac{1}{2}\lambda_m^3\left(x\right) \tag{3.3}$$

and,

$$M_0=\frac{1}{2}\left[M_0M_0^{\dagger}+trM_0M_0^{\dagger}\mathbf{1}\right]M_0 \tag{3.4}$$

Let us now consider equation (3.4) and assume M_0 to be real. We diagonalize the equation through unitary matrices V_1 and V_2 to obtain

$$M_{diag}=V_1^{\dagger}M_0V_2 \tag{3.5}$$

$$M_{diag}=\frac{1}{2}\left[M_{diag}^2+trM_{diag}^2\mathbf{1}\right]M_{diag} \tag{3.6}$$

we express the above relation in terms of the eigenvalues

$$M_{diag}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \tag{3.7}$$

then we obtain the following relations

$$\lambda_1=\frac{1}{2}\left[2\lambda_1^3+\lambda_1\left(\lambda_2^2+\lambda_3^2\right)\right] \tag{3.8a}$$

$$\lambda_2=\frac{1}{2}\left[2\lambda_2^3+\lambda_2\left(\lambda_3^2+\lambda_1^2\right)\right] \tag{3.8b}$$

$$\lambda_3 = \frac{1}{2}\left[2\lambda_3^3 + \lambda_3\left(\lambda_1^2 + \lambda_2^2\right)\right] \tag{3.8c}$$

We assume the eigenvalues to be real and order them in the sequence $\lambda_1 \leq \lambda_2 \leq \lambda_3$,and consider only positive values for these fermions. There are then the following four possibilities

(i) All the λ_is are zero. This is a trivial solution

(ii) All the λ_is are non-zero in which case they must all be equal. This is also a trivial solution if $V_1 = V_2$, as is the case for U_0 and D_0 defined in (1.11) and (1.12) and, when M_0 becomes a unit matrix

(iii) $\lambda_1 = 0$, in which case $\lambda_2 = \lambda_3$

(iv) $\lambda_1 = \lambda_2 = 0$, in which case $\lambda_3 = 1$

Therefore, there are only two non-trivial solution, which we describe as follows

(i) "Hierarchical" solution with $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$

$$M_{diag}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

$$M_0^{(1)} = V_1 M_{diag}^{(1)} V_2^\dagger \tag{3.10}$$

(ii) "Semi-hierarchical" solution with $\lambda_1 = 0, \lambda_2 = \lambda_3 = \sqrt{\frac{2}{3}}$

$$M_{diag}^{(2)} = \sqrt{\frac{2}{3}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.11}$$

$$M_0^{(2)} = V_1 M_{diag}^{(2)} V_2^\dagger \tag{3.12}$$

A. Hierarchical solutions

In Appendix I a general form of $M_0^{(1)}$ is obtained by using equation (3.4) directly.

Below we obtain $M_0^{(1)}$,assumed real and symmetric, from relation (3.5) by taking

$$V_1 = V_2 = V \tag{3.13}$$

where we will take V as the product of three rotations that diagonalizes, in succession, the 2-3, 1-3 and 1-2 sub-matrices by angles θ_1,θ_2,and θ_3,respectively i.e.

$$V = \begin{bmatrix} c_2c_3 & c_2s_3 & s_2 \\ -c_1s_3 - s_1s_2c_3 & c_1c_3 - s_1s_2s_3 & s_1c_2 \\ s_1s_3 - c_1s_2c_3 & -s_1c_3 - c_1s_2s_3 & c_1c_2 \end{bmatrix} \tag{3.14}$$

where s_i and c_i are the corresponding sine and cosine-values. With (3.13) and (3.14) we then have the following simple form

$$M_0^{(1)} = \begin{bmatrix} s_2^2 & c_2s_1s_2 & c_1c_2s_2 \\ c_2s_1s_2 & c_2^2s_1^2 & c_1c_2^2s_1 \\ c_1c_2s_2 & c_1c_2^2s_1 & c_1^2c_2^2 \end{bmatrix} \tag{3.15}$$

There is thus a direct connection between the mass matrix and the rotation parameters. This shows a classsic hierarchy pattern, particularly for the case of small angles, in which case one can write ($c_i \approx 1$)

$$M_0^{(1)} = \begin{bmatrix} s_2^2 & s_1s_2 & s_2 \\ s_1s_2 & s_1^2 & s_1 \\ s_2 & s_1 & 1 \end{bmatrix} \tag{3.16}$$

B. Semi-hierarchical solutions

Following the same procedure as above, the semi-hierarchical patterns are created by writing

$$M_0^{(2)} = VM_{diag}^{(2)}V^\dagger \tag{3.17}$$

With (3.13) and (3.14), $M_0^{(2)}$ in (3.12) becomes, for small angles ($c_i \approx 1$)

$$M_0^{(2)} = \sqrt{\frac{2}{3}} \begin{bmatrix} s_3^2 & s_3 & s_2 - s_3s_1 \\ s_3 & s_1^2+1 & -s_2s_3 \\ s_2 - s_3s_1 & -s_2s_3 & s_1^2+1 \end{bmatrix} \tag{3.18}$$

IV. U and D matrices, and the CKM angles

A. U-matrix

If we assume $\lambda_b << \lambda_t$ then the contribution of the second term in (2.4) given by

$$\frac{1}{12}DD^+U \tag{4.1}$$

will be of the order $\frac{1}{12}\lambda_b^2\lambda_t$ which is much smaller than the first term in (2.4) which is of the order$\frac{1}{2}\lambda_t^3$. It can, therefore, be safely neglected. The U-matrix will satisfy, to an excellent approximation, the following

$$-\frac{dU}{dx} = \frac{1}{4}\left[UU^+ + Tr(UU^+)\mathbf{1}\right]U \tag{4.2}$$

which is the same equation as (3.4) for M_0 and will have the solutions already discussed.

We now assume that U_0 is of the "hierarchical form" so that

$$U_{diag} = M^{(1)}_{diag} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

and

$$U_0 = V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^{\dagger} \tag{4.4}$$

If we take V to be the same as the CKM matrix

$$V \;\; = V_{CKM} \tag{4.5}$$

then, as discussed in III A, equations (3.15), we have

$$U_0 = M^{(1)}_0 = \begin{bmatrix} s_2^2 & c_2 s_1 s_2 & c_1 c_2 s_2 \\ c_2 s_1 s_2 & c_2^2 s_1^2 & c_1 c_2^2 s_1 \\ c_1 c_2 s_2 & c_1 c_2^2 s_1 & c_1^2 c_2^2 \end{bmatrix} \tag{4.6}$$

where the angles now are the same as given by the CKM matrix , we call it "input" CKM, giving us the expression (with $c_i \approx 1$)

$$U_0 = \begin{bmatrix} s_2^2 & s_1 s_2 & s_2 \\ s_1 s_2 & s_1^2 & s_1 \\ s_2 & s_1 & 1 \end{bmatrix} \tag{4.7}$$

with

$$V^{(in)}_{CKM} = \begin{bmatrix} 1 & s_3 & s_2 \\ -s_3 & 1 & s_1 \\ s_1 s_3 - s_2 & -s_1 & 1 \end{bmatrix} \tag{4.8}$$

Let us elaborate further on the above result by expressing the CKM matrix in the Wolfenstein representation [2], keeping only the leading terms

$$V_{CKM} \;\; = \left[\begin{array}{ccc} 1 & \lambda & A\lambda^3 \left(\rho - i\eta\right) \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3 \left(1 - \rho - i\eta\right) & -A\lambda^2 & 1 \end{array} \right] \tag{4.9}$$

Comparing this with the expression for $V^{(in)}_{CKM}$ above we can extract s_1, s_2, and s_3 as follows considering only the magnitudes of the Wolfenstein parameters (ignoring phases).

$$s_1 \approx A\lambda^2 \tag{4.10a}$$

$$s_2 \approx A\lambda^3 \sqrt{\rho^2 + \eta^2} \tag{4.10b}$$

$$s_3 \approx \lambda \tag{4.10c}$$

Substituting the above values in the expression for U_0 given by we obtain

$$U_0 = \begin{bmatrix} A^2\lambda^6\left(\rho^2+\eta^2\right) & A^2\lambda^5\sqrt{\rho^2+\eta^2} & A\lambda^3\sqrt{\rho^2+\eta^2} \\ A^2\lambda^5\sqrt{\rho^2+\eta^2} & A^2\lambda^4 & A\lambda^2 \\ A\lambda^3\sqrt{\rho^2+\eta^2} & A\lambda^2 & 1 \end{bmatrix} \tag{4.11}$$

Experimentally [22], one finds, in order of magnitude terms,

$$A \approx 1 \tag{4.12a}$$

$$\sqrt{\rho^2+\eta^2} \approx \lambda \tag{4.12b}$$

therefore, the above expression simplifies to

$$U_0 \approx \begin{bmatrix} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{bmatrix} \tag{4.13}$$

which is in excellent agreement with the behavior one expects for U_0 [1].

To recapulate then, after including the x-dependence from (2.3), we have the complete expression for U given by

$$U = \begin{bmatrix} s_2^2 & s_1 s_2 & s_2 \\ s_1 s_2 & s_1^2 & s_1 \\ s_2 & s_1 & 1 \end{bmatrix} \lambda_{0t}\left[1+x\lambda_{0t}^2\right]^{-\frac{1}{2}} \tag{4.14}$$

where s_i are the "input" CKM parameters.

The eigenvalues of U_0 are, of course, the same as $M_{diag}^{(1)}$ i.e $(0,0,1)$. To obtain the correct values, as we stated in the Introduction, one must go beyond the standard model which we propose to do in Section V by incorporating texture zeros.

B. D-matrix

We assume here that D is different from U by the fact that the solution of the "uncoupled" part

$$-\frac{dD}{dx} = \frac{1}{4}\left[DD^+ + Tr(DD^+)\mathbf{1}\right] D \tag{4.15}$$

is now given by $M_0^{(2)}$,the "semi-hierarchical" solution (3.11) and (3.12) with $V_1 = V_2 = V$

$$M_{diag}^{(2)} = \sqrt{\frac{2}{3}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_0^{(2)} = V M_{diag}^{(2)} V^\dagger$$

However, this can not be a complete solution for the D-equation given by (2.5)

$$-\frac{dD}{dx} = \frac{1}{4}\left[DD^+ + Tr(DD^+)\mathbf{1}\right] D + \frac{1}{12} UU^+ D$$

since the coupling term neglected in (4.15)

$$\frac{1}{12} UU^+ D \tag{4.16}$$

can be very large because $\lambda_t >> \lambda_b$. This is in stark contrast to U where the coupling term was negligible.

If we consider the x-independent (scale-independent) coupling term in (4.16), then we have

$$\frac{1}{12} U_0 U_0^+ D_0 = \frac{1}{12} M_0^{(1)} M_0^{(1)} M_0^{(2)} \tag{4.17}$$

$$= \frac{1}{12}\sqrt{\frac{2}{3}} \left[V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^\dagger \right] \left[V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^\dagger \right] \left[V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^\dagger \right] \tag{4.18}$$

Therefore,

$$\frac{1}{12} U_0 U_0^+ D_0 = \frac{1}{12}\sqrt{\frac{2}{3}} \left[V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^\dagger \right] = \frac{1}{12}\sqrt{\frac{2}{3}} M_0^{(1)} \tag{4.19}$$

So there is a contribution from $M_0^{(1)}$ in the D-matrix that needs to be added to $M_0^{(2)}$. One can, to a good approximation, write the differential equation for D in (2.5) as

$$-\frac{dD}{dx} = V \left[\frac{1}{2}\lambda_b^3 \sqrt{\frac{2}{3}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{12}\sqrt{\frac{2}{3}} \lambda_t^2 \lambda_b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] V^\dagger \tag{4.20}$$

$$= \frac{1}{2}\lambda_b^3 M_0^{(2)} + \frac{1}{12}\sqrt{\frac{2}{3}} \lambda_t^2 \lambda_b M_0^{(1)} \tag{4.21}$$

We can solve for the 33-elements on both sides of the equation (4.21), assumed dominated by λ_b and λ_t, and obtain

$$-\frac{d\lambda_b}{dx} = \frac{1}{2}\lambda_b^3 + \frac{1}{12}\lambda_t^2 \lambda_b \tag{4.22}$$

An exact analytical solution of (4.22) is given in Appendix II

It is important to note that, as far as the 22-elements of the two diagonal matrices in (4.20) are concerned, the second matrix is a "small perturbation" on the first because it is, in fact, negligible (zero) compared to the first . In contrast, with respect to the 33-matrix elements, it is the first matrix which is a "small perturbation" since $\lambda_b << \lambda_t$.Furthermore, we note that the x-dependence of the 22-diagonal element will not be the same as that of the 33-element, which is inconsistent with our assumption that the entire D matix is represented by a single function of x.

However, since the first matrix is extremely small we can write an approximate solution for D as follows

$$D_{diag} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.23}$$

$$D_0 = V \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} V^\dagger \tag{4.24}$$

ϵ is assumed to be a constant, independent of x, and where in the 33-element of D_{diag} we have ignored the contribution of ϵ compared to 1.

To determine ϵ one can (numerically) integrate the first term in (4.22), $\frac{1}{2}\lambda_b^3$, from the knowledge of the analytic expression for λ_b given in Appendix II, and compare it to the integral of the second term, $\frac{1}{12}\lambda_t^2\lambda_b$, with λ_t given in (2.3). The ratio of the two will estimate the value of ϵ. If we take the ratio of the Yukawa couplings λ_{0b} and λ_{0t} to be the same as the ratio of the masses then we obtain $\epsilon \approx .01$ which is found to be essentially independent of x.

Another, rough order of magnitude, estimate of ϵ can also be obtained on the basis of the ratio of first to the second term in (2.5) above

$$\epsilon \approx \left(\frac{\lambda_b^3}{\lambda_t^2 \lambda_b} \right)^{\frac{1}{2}} = \frac{m_b}{m_t} \approx .025 \tag{4.25}$$

The square root is taken because, effectively, the first term in the equation (2.5) for $\dfrac{dD}{dx}$ is proprtional to D^3 while the second is proprtional to $U^2 D$, so the ratio is $\approx D^2$. Therefore, the contribution to D will involve the square root. The numerical value of ϵ from (4.25) **is** obtained by taking the ratio of the Yukawa couplings to be the same as the ratio of the corresponding masses.

In our calculations to follow we will take ϵ as an arbitrary parameter keeping in mind, however, that its order of magnitude is $\approx 10^{-2}$.

To obtain an approximate solution for λ_b,we can ignore the first term in (4.22), which is very small, to write, after substituting expression (2.3) for λ_t ,

$$-\frac{d\lambda_b}{dx} \approx \frac{1}{12}\lambda_t^2\lambda_b \ = \frac{1}{12}\lambda_{0t}^2 \left[1 + x\lambda_{0t}^2\right]^{-1} \lambda \tag{4.26}$$

the solution to which is

$$\lambda_b \left(x\right) = \lambda_{0b} \left[1 + x\lambda_{0t}^2\right]^{-\frac{1}{12}} \tag{4.27}$$

$\lambda_{0b} = \lambda_b \left(t_0\right) = \lambda_b \left(x = 0\right)$

From (4.24) we can write down the expression for D

$$D = \begin{bmatrix} \epsilon s_3^2 & \epsilon s_3 & s_2 - \epsilon s_1 s_3 \\ \epsilon s_3 & s_1^2 + \epsilon & s_1 \\ s_2 - \epsilon s_1 s_3 & s_1 & 1 \end{bmatrix} \lambda_{0b} \left[1 + x\lambda_{0t}^2\right]^{-\frac{1}{12}} \tag{4.28}$$

where s_i's are the "input" CKM parameters. We can express s_i in terms of λ, as we did previously when we considered U_0, to obtain

$$D_0 \approx \begin{bmatrix} \epsilon\lambda^2 & \epsilon\lambda & \lambda^4 - \epsilon\lambda^3 \\ \epsilon\lambda & \lambda^4 + \epsilon & \lambda^2 \\ \lambda^4 - \epsilon\lambda^3 & \lambda^2 & 1 \end{bmatrix} \tag{4.29}$$

The estimate for ϵ given by (4.25) is, numerically $\approx \lambda^2$, and, correspondingly we have

$$D_0 \approx \begin{bmatrix} \lambda^4 & \lambda^3 & \lambda^4 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{bmatrix} \tag{4.30}$$

This expression shows that the hierarchy in D_0 is not as pronounced as it was found in U, which is consistent with generally accepted form for D_0 in [1]

$$D_0 \approx \begin{bmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{bmatrix} \tag{4.31}$$

Our results are in excellent agreement with the above if we ignore the slight discrepancy in the 13- and 31- components which, in fact, are quite inconsequential as we will see below when we discuss texture zeros.

The eigenvalues of D_0 are given by the eigenvalues of D_{diag}, i.e. $(0, \epsilon, 1)$. As pointed out in the Introduction, to obtain the correct values one must go beyond the standard model which we propose to do in Section V by incorporating texture zeros.

V. Mass eigenvalues and CKM matrix in the quark sector with texture zeros

As stated in the Introduction the most acceptable structures with texture zeros are the following

$$U_0 = \begin{bmatrix} 0 & 0 & X \\ 0 & X & 0 \\ X & 0 & X \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{bmatrix}$$

In terms of our results in section IV we then have

$$U_0 = \begin{bmatrix} 0 & 0 & s_2 \\ 0 & s_1^2 & 0 \\ s_2 & 0 & 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & \epsilon s_3 & 0 \\ \epsilon s_3 & s_1^2+\epsilon & s_1 \\ 0 & s_1 & 1 \end{bmatrix} \tag{5.1}$$

A. CKM matrix

The matrix elements above involve the "input" CKM parameters s_i already defined (4.8)

$$V_{CKM}^{(in)} = \begin{bmatrix} 1 & s_3 & s_2 \\ -s_3 & 1 & s_1 \\ s_1 s_3 - s_2 & -s_1 & 1 \end{bmatrix} \tag{5.2}$$

The numerical values of the parameters are known from experiments [22]and are given by

$$s_1 = (.038 - .044)\,, s_2 = (.0025 - .0048)\,, s_3 = (.219 - .226\ ) \tag{5.3}$$

The "output" CKM is given by the standard definition

$$V_{CKM}^{(out)} = V_u^\dagger V_d \tag{5.4}$$

V_d is the unitary matrix which diagonalizes D_0; and V_u is the unitary matrix which diagonalizes U_0. In terms of the rotation angles it is given by (assuming $c_i' \approx 1$)

$$V_{CKM}^{(out)} = \begin{bmatrix} 1 & s_3' & s_2' \\ -s_3' & 1 & s_1' \\ s_1' s_3' - s_2' & -s_1' & 1 \end{bmatrix} \tag{5.5}$$

primes are used to distinguish from the "input" case.

It is now easy to observe from the expressions in (5.1) that the rotation for diagonalizing the 23-sub matrix involves only D_0, and for the 13-sub matrix it involves only U_0. Therefore, by the nature of the location of the texture zeros

a (complete) mismatch is achieved between the rotation angles for U_0 and D_0 for these submatrices. We then have

$$s'_1 = s_1,\ s'_2 = s_2 \tag{5.6}$$

mismatch continues for the 12-submatrix, where the rotation only involves D_0 and we have

$$s'_3 = \frac{\epsilon s_3}{s_1^2 + \epsilon} \tag{5.7}$$

as we discussed in section IV the magnitude of ϵ was estimated to be $\approx 10^{-2}$ which is much larger than $s_1^2\,(\approx .0016)$ .Therefore,

$$s'_3 \approx s_3 \tag{5.8}$$

we conclude that

$$V_{CKM}^{(out)} = V_{CKM}^{(in)} \tag{5.9}$$

B. Mass eigenvlues

Plugging in the following "input" values

$$s_1 = .04, s_2 = .004, s_3 = .22 \tag{5.10}$$

which are within the range of the experimental values given by (5.3) and taking $\epsilon = .05$, we obtain the following mass ratios, keeping in mind that the 33-matrix elements λ_t and λ_b are normalized to their observed mass values ("exp" means experimental values).

$$\frac{m_u}{m_t} = 4.8 \times 10^{-5}\ \left(\exp \approx 3 \times 10^{-5}\right)$$

$$\frac{m_c}{m_t} = 1.9 \times 10^{-3}\ \left(\exp \approx 8 \times 10^{-3}\right)$$

$$\frac{m_d}{m_b} = 2.4 \times 10^{-3}\ \left(\exp \approx 2 \times 10^{-3}\right)$$

$$\frac{m_s}{m_b} = 5.5 \times 10^{-2}\ \left(\exp \approx 3 \times 10^{-2}\right) \tag{5.11}$$

The agreement with experiments is very good considering the fact that no attempt was made to do a detailed numerical analysis to fit all the data.

VI. Solutions to the RGEs in the lepton sector

The RG equations we wish to solve for N and E are given by (2.6) and (2.7). We notice that the last term in each equation involves a trace which multiplies equally all the matrix elements of N and E. Therefore, it will effectively change the scale, though we realize that this is not completely correct since there are also non-linear terms present in the equation. We will ignore the last terms in (2.6) and (2.7), nevertheless, as a simplifying assumption. We then have the following

$$-\frac{dE}{dx} = \frac{1}{12}\left[3EE^{+} + Tr(EE^{+})\mathbf{1}\right]E \;+\frac{1}{12}NN^{+}E \tag{6.1}$$

$$-\frac{dN}{dx} = \frac{1}{12}\left[3NN^{+} + Tr(NN^{+})\mathbf{1}\right]N \;+\frac{1}{12}EE^{+}N \tag{6.2}$$

solving these equations we want to follow, as closely as possible, the analogy to U and D, in particular to identify N with U, and E with D.

Considering first, as we did for U and D, the uncoupled terms in the square brackets. We find that they satisfy an equation very similar to (3.4), except for the numerical factors.

$$-\frac{dL}{dx} = \frac{1}{12}\left[3LL^{\dagger} + Tr\left(LL^{\dagger}\right)\mathbf{1}\right]L \tag{6.3}$$

Following the same procedure as before, we write

$$L = L_0\lambda_l\left(x\right) \tag{6.4}$$

$\lambda_l\left(x\right)$ represents the dominant 33-matrix element, and,

$$L_0 = \frac{1}{4}\left[3L_0L_0^{\dagger} + trL_0L_0^{\dagger}\mathbf{1}\right]L_0 \tag{6.5}$$

$$L_{diag} = V_1^{\dagger}L_0V_2 \tag{6.6}$$

$$L_{diag} = \frac{1}{4}\left[3L_{diag}^2 + trL_{diag}^2\mathbf{1}\right]L_{diag} \tag{6.7}$$

We express the above relation in terms of the eigenvalues

$$L_{diag} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \tag{6.8}$$

then we obtain the following relations

$$\lambda_1 = \frac{1}{4}\left[4\lambda_1^3 + \lambda_1\left(\lambda_2^2 + \lambda_3^2\right)\right] \tag{6.9a}$$

$$\lambda_2 = \frac{1}{4}\left[4\lambda_2^3 + \lambda_2\left(\lambda_3^2 + \lambda_1^2\right)\right] \tag{6.9b}$$

$$\lambda_3 = \frac{1}{4}\left[4\lambda_3^3 + \lambda_3\left(\lambda_1^2 + \lambda_2^2\right)\right] \tag{6.9c}$$

Once again we have only two non-trivial solutions.

(i) "Hierarchical" solution with $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$

$$L_{diag}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.10}$$

$$L_0^{(1)} = V_1 L_{diag}^{(1)} V_2^\dagger \tag{6.11}$$

(ii) "Semi-hierarchical" solution with $\lambda_1 = 0, \lambda_2 = \lambda_3 = \sqrt{\frac{4}{5}}$

$$L_{diag}^{(2)} = \sqrt{\frac{4}{5}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.12}$$

$$L_0^{(2)} = V_1 L_{diag}^{(2)} V_2^\dagger \tag{6.13}$$

We now make the following important identifications:

(i) As we mentioned earlier, we identify N with U so that we take it to be of the hierchical type i.e.

$$N_{diag} = L_{diag}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.14}$$

(ii) We, therefore, take the 33-element of N much larger than that of E, which implies that

$$\lambda_{3v} >> \lambda_\tau \tag{6.15}$$

(iii) We observe that unlike U, D, E, which are entirely standard model constituents, N couples a standard model object, the left-handed neutrino, ν_L, to a particle not in the standard model framework. This leads us to identify V_1 as the small angle rotation matrix of the standard model i.e. V_{CKM},as we did for U, D and E and we assume that V_2 represents the large-angle mixing matrix of the neutrino data i.e. the analog of the CKM matrix **[10]**, we have called it V_{large}. Therefore,

$$V_1 = V_{CKM} \tag{6.16}$$

$$V_2 = V_{large} \tag{6.17}$$

from (6.11)

$$N_0 = V_{CKM} L^{(1)}_{diag} V^{\dagger}_{large} = V_{CKM} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^{\dagger}_{large} \tag{6.18}$$

$$N = N_0 \lambda_{3v}(x) \tag{6.19a}$$

We can write down the expression for λ_{3v} in analogy to U taking account of the differences in the numerical factors in the equation for U in (2.4) and for N in (6.2),

$$\lambda_{3v}(x) = \lambda_{03v} \left[1 + x\lambda^2_{03v}\right]^{-\frac{1}{3}} \tag{6.19b}$$

where, $\lambda_{03v} = \lambda_{3v}(0)$.

(iv) We identify E with D, and write

$$E_{diag} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon' & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.20}$$

ϵ',as in the case of D, is the mixing parameter between the hierarchical and semi-hierarchical matrices. And, of course, since it is a standard model particle, we take, as we did for U_0 and D_0,

$$V_1 = V_2 = V_{CKM} \tag{6.21}$$

Therfore,

$$E_0 = V_{CKM} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon' & 0 \\ 0 & 0 & 1 \end{bmatrix} V^{\dagger}_{CKM} \tag{6.22a}$$

$$E \ = E_0\lambda_\tau(x) \tag{6.22b}$$

Finally, for the Majorana neutrino mass matrix κ, described by the seesaw mechanism, as given by (1.5) we have

$$\kappa = N^T \left[M_R^{-1}\right] N \tag{6.23}$$

as mentioned in the Introduction, $\left[M_R^{-1}\right]$ is the distribution of the reciprocal of the seesaw neutrino masses. We then have from (1.5) and (6.18) and (6.19a) the following

$$\kappa = \left[V_{large} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V_{CKM}^\dagger \right] \left[M_R^{-1}\right] \left[V_{CKM} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V_{large}^\dagger \right] \lambda_{3\nu}^2 \tag{6.24}$$

We then obtain

$$\kappa \approx \kappa_0 \frac{\lambda_{3\nu}^2}{M_{33}} \tag{6.25}$$

$$\kappa_0 = V_{large} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V_{large}^\dagger \tag{6.26}$$

where $\left(\dfrac{1}{M_{33}}\right)$ is the 33-matrix element of $\left[M_R^{-1}\right]$, and where we have used the CKM properties in the product in (6.24), namely small
gles, implying $c_i \approx 1$. The rotation matrix V_{large} is the CKM-analog [10] given by

$$V_{large} = \begin{bmatrix} C_2C_3 & C_2S_3 & S_2 \\ -C_1S_3 - S_1S_2C_3 & C_1C_3 - S_1S_2S_3 & S_1C_2 \\ S_1S_3 - C_1S_2C_3 & -S_1C_3 - C_1S_2S_3 & C_1C_2 \end{bmatrix} \tag{6.27}$$

where, in order to distinguish it from the CKM matrix, we have used capital letters.

Therefore, the neutrino mass matrix, κ_0 is given by

$$\kappa_0 = \begin{bmatrix} S_2^2 & C_2S_1S_2 & C_1C_2S_2 \\ C_2S_1S_2 & C_2^2S_1^2 & C_1C_2^2S_1 \\ C_1C_2S_2 & C_1C_2^2S_1 & C_1^2C_2^2 \end{bmatrix} \tag{6.28}$$

which is similar to U except for the presence of large angles, and the seesaw contribution.

Needless to say, the parameters of V_{large} in (6.27) and (6.28) are what we have defined as "input" parameters.

VII. Mass eigenvalues and mixing angles in the lepton sector with texture zeros

For the charged leptons, as we mentioned earlier, the structure of E_0 is assumed to be the same as that of D_0

$$E_0 = \begin{bmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{bmatrix} \tag{7.1}$$

for κ_0, as stated in the Introduction, we have two types of structures[18].

(a) A-type structure [18]

$$\kappa_0 = \begin{bmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{bmatrix} \quad or\ \kappa_0 = \begin{bmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{bmatrix} \tag{7.2}$$

matrices give hierarchical neutrino mass values. We will only consider the first matrix above since the second matrix gives the same results [18,19].

(b) C-type structure [18]

$$\kappa_0 = \begin{bmatrix} X & X & X \\ X & 0 & X \\ X & X & 0 \end{bmatrix} \tag{7.3}$$

gives an inverted hierarchy [18,19].

In the following we will not consider the C-type structure for a simple reason. Our entire construction of the Yukawa matrices has had at its basis the "hierarchical" and "semi-hierarchical" primordial systems which clearly can not be compatible with inverted hierarchy. A quick calculation confirms this conclusion.

We will revisit the C-type structure in section VIII.

A. Mass eigenvalues for charged leptons

We simply follow the results from the D-matrix, and write

$$E_0 = \begin{bmatrix} 0 & \epsilon' s_3 & 0 \\ \epsilon' s_3 & s_1^2 + \epsilon' & s_1 \\ 0 & s_1 & 1 \end{bmatrix} \tag{7.4}$$

the angles s_i are the usual "input" CKM angles, and ϵ' is the analog of ϵ

in D_0 which is the mixing parameter between the "hierarchical" and "semi-hierarchical" representations. For the "input" CKM values given in (5.10) and $\epsilon' = .02$, we find

$$\frac{m_\mu}{m_\tau} = 2.1 \times 10^{-2} \ \left(\exp \approx 6 \times 10^{-2}\right) \tag{7.5a}$$

$$\frac{m_e}{m_\tau} = 9.2 \times 10^{-4} \ \left(\exp \approx 3 \times 10^{-4}\right) \tag{7.5b}$$

"exp" means experimental values [22]. Once again, there is a good agreement with experiments.

B. Mixing angles and mass eigenvalues for Majorana neutrinos

The A-type Majorana matrix, κ_0, has texture zeros as given by [18,19].

$$\kappa_0 = \begin{bmatrix} 0 & 0 & C_1C_2S_2 \\ 0 & C_2^2S_1^2 & C_1C_2^2S_1 \\ C_1C_2S_2 & C_1C_2^2S_1 & C_1^2C_2^2 \end{bmatrix} \tag{7.6}$$

where C_i and S_i are the parameters of the "input" CKM-analog, already defined in (6.27)

$$V_{large}^{(in)} = \begin{bmatrix} C_2C_3 & C_2S_3 & S_2 \\ -C_1S_3 - S_1S_2C_3 & C_1C_3 - S_1S_2S_3 & S_1C_2 \\ S_1S_3 - C_1S_2C_3 & -S_1C_3 - C_1S_2S_3 & C_1C_2 \end{bmatrix} \tag{7.7}$$

This matrix is defined in the basis when E_0 is diagonal. The experimental values of the parameters are [3,4,5,6,7,8]

$$S_1 = (.54 - .83)\,; S_3 = (.40 - .70)\,; S_2 \leq .16 \tag{7.8}$$

The matrix that will diagonalize κ_0 in (6.28) is

$$\begin{bmatrix} C_2'C_3' & C_2'S_3' & S_2' \\ -C_1'S_3' - S_1'S_2'C_3' & C_1'C_3' - S_1'S_2'S_3' & S_1'C_2' \\ S_1'S_3' - C_1'S_2'C_3' & -S_1'C_3' - C_1'S_2'S_3' & C_1'C_2' \end{bmatrix} \tag{7.9}$$

However, this matrix is not exactly the CKM-analog [10], $V_{large}^{(out)}$.We need to include rotations that diagonalize E_0 as well, since the CKM-analog is defined in the mass basis of E_0. We note, however, that the rotation angles to diagonalize the 2-3 and 1-3 sub matrices of E_0 are very small and can be ignored. We will only consider the 1-2 submatrix in E_0 since, here, for ϵ' of the order .02, needed to give the correct charged lepton masses, as discussed earlier, the s_3 parameter turns out to be $\approx .22$ [22]. Thus we define

$$V_{large}^{(out)} = \begin{bmatrix} C_2'C_3' & C_2'S_3' & S_2' \\ -C_1'S_3' - S_1'S_2'C_3' & C_1'C_3' - S_1'S_2'S_3' & S_1'C_2' \\ S_1'S_3' - C_1'S_2'C_3' & -S_1'C_3' - C_1'S_2'S_3' & C_1'C_2' \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.10}$$

The effect of the second bracket above is to change only the angle θ'_3, by θ_{3E},where $sin\theta_{3E} = s_3$ ($\approx .22$). Thus

$$\theta_1^{(out)} = \theta'_1,\ \theta_2^{(out)} = \theta'_2,\ \theta_3^{(out)} = \theta'_3 - \theta_{3E} \tag{7.11}$$

First of all a simple examination of the structure of the 23-sub matrix of κ_0 in (7.6) shows that the angle of rotation, θ'_1, that would diagonalize κ_0 is precisely $\theta_1$itself,

$$S'_1 = S_1 \tag{7.12}$$

actual value we take for S_1 will be determined when we calculate the neutrino masses.

For the other two angles we find

$$S'_2 \approx C_1^2 S_2 \tag{7.13}$$

where we take $C_2 = C'_2 = 1$ since S_2 is small. And for S'_3,we obtain the following equation

$$T_3'^2 - S_2 \left(\frac{C_1^3}{S_1} \right) T'_3 - 1 = 0 \tag{7.14}$$

$T'_3 = tan\theta'_3 = \frac{S'_3}{C'_3}$.

The diagonalized form for the full mass matrix κ, defined in terms of κ_0 in (6.25) and (7.6), is found to be the following (again we take $C_2 = C'_2 = 1$)

$$\frac{\lambda_{03\nu}^2}{M_{33}} \begin{bmatrix} \left(\frac{C_1 S_1 S_2}{T'_3} \right) & 0 & 0 \\ 0 & -T'_3 C_1 S_1 S_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.15}$$

We now compare the above mass values with the neutrino oscillation data which give the following mass-squared differences[3,4,5,6,7,8]

$$\Delta m_{12}^2 = (2 - 50) \times 10^{-5} eV^2,\ \Delta m_{23}^2 = (1.2 - 5) \times 10^{-3} eV^2 \tag{7.16}$$

The mixing angle values that are consistent with the experimental values given in (7.8) and (7.16) are found to be

$$S_2 = .16,\ S_1 = .54 \tag{7.17}$$

With these parameters and equation (7.14) we obtain

$$S'_3 = 0.74 \tag{7.18}$$

and from the relation between $\theta_3^{(out)}$, θ_3' and θ_{3E} given in (7.11) we obtain

$$S_3^{(out)} = .57 \tag{7.19}$$

The mass values for κ_0 are determined from the above parameters and by taking

$$\frac{m_{3\nu}^2}{M_{33}} = 7.1 \times 10^{-2} eV \tag{7.20}$$

where $m_{3\nu}$ is the dominant 33-component of the (Dirac) neutrino Yukawa matrix, N. We then obtain

$$m_1 = 4.8 \times 10^{-3} eV \tag{7.21a}$$

$$m_2 = 5.8 \times 10^{-3} eV \tag{7.21b}$$

$$m_3 = 7.1 \times 10^{-2} eV \tag{7.21c}$$

These values are consisitent with the experimental results (7.8).

As for comparing the "input" and "output" values of the mixing matrices, we have

$$S_1^{(out)} = S_1^{(in)} = .54 \tag{7.22a}$$

$$S_2^{(out)} = .11,\ S_2^{(in)} = .16 \tag{7.22b}$$

$$S_3^{(out)} = S_3^{(in)} = .57 \tag{7.22c}$$

We point out that through (7.9) and (7.10) we first obtained $S_3^{(out)}$ and then simply assumed $S_3^{(in)}$ to have the same value since $S_3 \left(= S_3^{(in)}\right)$ was not involved in constructing the mass matrix (7.6) for κ_0 and, therefore, there were no constraints on its value.

The above predictions are consistent with experiments. Moreover, apart from the differences in S_2,which are minor, the "input" and "output" mixing parameters are the same.

We also point out that, in order to get the correct neutrino masses, the value of S_2 needs to be large, as already noted for the case of A-type structure in ref.[18] and [19]. As for the possibility of having a maximal coupling in the 23 sector of κ_0 (i.e. $S_1 = \frac{1}{\sqrt{2}} = .71$) we opted instead to reconcile the "input" and "output" values of S_2 which implied, through relation (7.13), that we have as large a C_1 as possible, within experimental bounds, which led us to the value of S_1 given by (7.17).

We can estimate the mass, M_{33}, for the seesaw neutrino in (7.20) by following our assumption that N and E have mass properties that are similar to U and D respectively, in which case

$$\frac{m_{3\nu}}{m_\tau} \approx \frac{m_t}{m_b} \tag{7.23}$$

and, therefore, putting in the other known mass values, we get $m_{3\nu}^2 \approx 10^3 GeV^2$. From (7.20) we then find

$$M_{33} \approx 10^{13} GeV \tag{7.24}$$

VIII. Inverted Hierarchy in Neutrinos, revisited

One might ask if in the eigenvalue equation for the leptons (6.9a,b,c), we could have chosen the following "primordial" solutions to generate an inverted hierarchy for the neutrinos, leaving the charged leptons in the hierarchical pattern

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0 (neutrinos) \tag{8.1a}$$

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 1 \quad \mathit{or} \quad \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 1\ (\mathit{charged\ leptons}) \tag{8.1b}$$

First of all, we notice immediately that for the charged lepton RGE (6.1) the coupling term

$$\frac{1}{12} N N^+ E \tag{8.2}$$

will vanish as it involves the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.3}$$

In the absence of the coupling contribution, the charged lepton matrix will have two choices, instead of the previous situation when it was, like D_0, a mixture of "hierarchical" and "semi-hierarchical" bases. Either it has to be strictly "semi-hierarchical", which will be inconsistent with experiments since it would mean two mass eigenvalues very close to each other. Or it has to be "hierarchical" which would mean it would be like the up-quark matrix, U_0, which would, however, give mass ratios an order of magnitude smaller than observed. Thus this model will not work for the charged leptons. We will proceed anyway, with the hope that there may be another, totally different, scenario possible for it, and take the charged lepton matrix to be diagonal so its parameters do not enter into the calculation for neutrinos.

The neutrino mass matrix κ_0 will have the following expression (before incorporating texture zeros)

$$\kappa_0 = V_{large} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V_{large}^{\dagger} \tag{8.4}$$

$$\kappa \approx \kappa_0 \frac{\lambda_{1\nu}^2}{M_{11}} \tag{8.5}$$

with V_{large} as the "input" matrix as before, and $\left(\dfrac{1}{M_{11}}\right)$ the 11-element of $\left[M_R^{-1}\right]$.

Instead of writing a long and complicated matrix that will result from the above product we write it after the texture zeros, of the C-variety, are already imposed. At the same time, we take account of the fact, evident from κ_0's structure given in (7.3), that with vanishing 22 and 33 diagonal elements, the rotation angle for the matrix diagonalizing the 23-sub matrix will be 45^0. Thus, the "output" S_1 will automatically be

$$S_1^{(out)} = \frac{1}{\sqrt{2}} \tag{8.6}$$

the above value is, indeed, allowed by the experiments we also take

$$S_1^{(in)} = \frac{1}{\sqrt{2}} \tag{8.7}$$

further simplication occurs because of (8.6), namely [19]

$$S_2^{(out)} = 0 \tag{8.8}$$

and, here again, we take the same value for the "input" parameter

$$S_2^{(in)} = 0 \tag{8.9}$$

since it is allowed by the experiments.

Only S_3 now remains to be determined. With the above values of the "input" parameters we obtain the following, simplified expression,

$$\kappa_0 = \begin{bmatrix} C_3^2 & -\dfrac{C_3 S_3}{\sqrt{2}} & \dfrac{C_3 S_3}{\sqrt{2}} \\ -\dfrac{C_3 S_3}{\sqrt{2}} & 0 & -\dfrac{S_3^2}{2} \\ \dfrac{C_3 S_3}{\sqrt{2}} & -\dfrac{S_3^2}{2} & 0 \end{bmatrix} \tag{8.10}$$

Diagonalizing it we obtain $\left(\text{with } T_3' = \frac{S_3'}{C_3'}\right)$

$$\begin{bmatrix} \dfrac{\left(C_3 S_3 + \frac{1}{2} T_3' S_3^2\right)}{T_3'} & 0 & 0 \\ 0 & \left(-T_3' C_3 S_3 + \frac{1}{2} S_3^2\right) & 0 \\ 0 & 0 & -\left(\frac{1}{2}\right) S_3^2 \end{bmatrix} \tag{8.11}$$

where T_3' satisfies

$$T_3'^2 + \left(\frac{C_3^2 - \frac{1}{2} S_3^2}{C_3 S_3}\right) T_3' - 1 = 0 \tag{8.12}$$

We choose $S_3\,(= S_3') = .58$, which is within the experimental range (7.8). From equation (8.12) we obtain, $S_3' = .51$ which is also the "output" value, since we have taken $\theta_{3E} = 0$, as explained above. Thus we have

$$S_1^{(out)} = S_1^{(in)} = \frac{1}{\sqrt{2}} \tag{8.13a}$$

$$S_2^{(out)} = S_2^{(in)} = 0 \tag{8.13b}$$

$$S_3^{(out)} = \ .51, \qquad S_3^{(in)} = .58 \tag{8.13c}$$

The "output" and "input" values are, therefore, identical for two angles and very close for the third.

With the above parameters and with

$$\frac{m_{1\nu}^2}{M_{11}} = 7.3 \times 10^{-2} eV \tag{8.14}$$

before, $m_{1\nu}^2 \approx 10^3 GeV^2$ and $M_{11} \approx 10^{13}\ GeV$) we have the following mass eigenvalues

$$m_1 = 7.1 \times 10^{-2} eV \tag{8.15a}$$

$$m_2 = 1.2 \times 10^{-2} eV \tag{8.15b}$$

$$m_3 = 8.3 \times 10^{-3} eV \tag{8.15c}$$

values are consistent with the experimental values given by (7.16)[3,4,5,6,7,8]

IX. Conclusion

One of our central assumptions was that the scale dependence of a Yukawa matrix is dictated entirely by the dominant 33-matrix element which can be factored out leaving behind a matrix which is independent of the scale. As a conseqence, the renormalization group equations for the Yukawa matrix can be expressed as two separate equations: one a differential equation for the 33-element and another an algebraic equation for the scale-independent 3x3 matrix.

It is the properties of the scale-independent matrices that has concerned us primarily. After constructing the solutions in terms of the mixing angles and incorporating texture zeros, consistent with hierarchical behavior, we made the following identifications for U, D, E, and (A-type) κ,

$$U = \begin{bmatrix} 0 & 0 & s_2 \\ 0 & s_1^2 & 0 \\ s_2 & 0 & 1 \end{bmatrix} \lambda_t, \qquad D = \begin{bmatrix} 0 & \epsilon s_3 & 0 \\ \epsilon s_3 & s_1^2 + \epsilon & s_1 \\ 0 & s_1 & 1 \end{bmatrix} \lambda_b$$

$$E = \begin{bmatrix} 0 & \epsilon' s_3 & 0 \\ \epsilon' s_3 & s_1^2 + \epsilon' & s_1 \\ 0 & s_1 & 1 \end{bmatrix} \lambda_\tau, \qquad \kappa = \begin{bmatrix} 0 & 0 & C_1 C_2 S_2 \\ 0 & C_2^2 S_1^2 & C_1 C_2^2 S_1 \\ C_1 C_2 S_2 & C_1 C_2^2 S_1 & C_1^2 C_2^2 \end{bmatrix} \frac{\lambda_{3\nu}^2}{M_{33}}$$

For the C-type neutrinos we have, as discussed in section VII, an especially simple mass matrix because of the location of the texture zeros which, automatically, result in $S_1 = \frac{1}{\sqrt{2}}$ and $S_2 = 0$, and give

$$\kappa = \begin{bmatrix} C_3^2 & -\dfrac{C_3 S_3}{\sqrt{2}} & \dfrac{C_3 S_3}{\sqrt{2}} \\ -\dfrac{C_3 S_3}{\sqrt{2}} & 0 & -\dfrac{S_3^2}{2} \\ \dfrac{C_3 S_3}{\sqrt{2}} & -\dfrac{S_3^2}{2} & 0 \end{bmatrix} \frac{\lambda_{1\nu}^2}{M_{11}}$$

which predicts an inverted hierarchy.

The above results provide simple expressions for the quark and lepton mass matrices in terms of the mixing angles. The manner in which we introduced the mixing parameters is self-consistent i.e. what we put in to construct the mass matrices (the "input") is recovered when we try to obtain the mixing matrices (the "output").

Another interesting result is that for the massive seesaw neutrinos only one mass-scale appears, M_{33} for normal hierarchy and M_{11} for inverted hierarchy. So the details of the seesaw mass distribution do not play a role, which is an important advantage in the model.

The texture zeros play a crucial role in determining the physical parameters in our model. It is well known that zeros in the mass matrices can arise through discrete symmetries [1, 11, 23, 24, 25, 26, 27, 28]. The precise group structure we need in order to reproduce the above texture zeros needs to be worked out and is being attempted currently [29].

To conclude, we have succeeded in obtaining a simple, transparent, and uniform framework to descibe four different pieces of data involving quarks and leptons. We assumed the Yukawa matrices to be real and symmetric with the dominant eigenvalue as an input. No attempt was made to do a detailed numerical analysis to fit the data but the matrices described above are found to give a very good descritipn of the mass eigenvalues and mixing matrices. Apart from the seesaw particles that are essential to the neutrinos, no new particles have been proposed–that is another big advantage to our model.

Acknowlegements

We wish to thank Dr. Rajasekaran for several discussions on this subject and Dr. D. P. Roy for his helpful comments. This work was supported in part by the U. S. Departmant of Energy under Grant No. DE-FG03-94ER40837

Appendix I

We notice from (3.9) and (3.10) that, because the determinant and trace are invariant, we have

$$det M_0^{(1)} = det M_{diag}^{(1)} = 0 \tag{A.1}$$

$$tr M_0^{(1)} = tr M_{diag}^{(1)} = 1 \tag{A.2}$$

for simplicity we will take $M_0^{(1)}$ to be real and symmetric. Using (A.2) in equation (3.4) we obtain

$$M_0^{(1)} = \frac{1}{2}\left[\left(M_0^{(1)}\right)^3 + M_0^{(1)}\right] \tag{A.3}$$

which we will rewrite as

$$\frac{1}{2}\left(\mathbf{1}+M_0^{(1)}\right)\left(\mathbf{1}-M_0^{(1)}\right)M_0^{(1)} = 0 \tag{A.4}$$

Out of the three factors above, only $\left(\mathbf{1}+M_0^{(1)}\right)$has an inverse since its determinant does not vanish,

$$det\left(\mathbf{1}+M_0^{(1)}\right) = det\left(\mathbf{1}+M_{diag}^{(1)}\right) \neq 0 \tag{A.5}$$

whereas the other two have vanishing determinants. Removing this term in (A.4) we obtain,

$$\left(\mathbf{1}-M_0^{(1)}\right)M_0^{(1)} = 0 \tag{A.6}$$

We note that a general 3x3 real, symmetric matrix with unit trace can be written as

$$M_0^{(1)} = (a+b+1)^{-1} \begin{bmatrix} a & d & \alpha \\ d & b & \beta \\ \alpha & \beta & 1 \end{bmatrix} \tag{A.7}$$

And, therefore,

$$\left(\mathbf{1} - M_0^{(1)}\right) = (a+b+1)^{-1} \begin{bmatrix} 1+b & -d & -\alpha \\ -d & 1+a & -\beta \\ -\alpha & -\beta & b+a \end{bmatrix} \tag{A.8}$$

From (A.6), (A.7) and (A.8) we have

$$\begin{bmatrix} 1+b & -d & -\alpha \\ -d & 1+a & -\beta \\ -\alpha & -\beta & b+a \end{bmatrix} \begin{bmatrix} a & d & \alpha \\ d & b & \beta \\ \alpha & \beta & 1 \end{bmatrix} = \tag{A.9}$$

$$\begin{bmatrix} -d^2 - \alpha^2 + a\,(b+1) & -bd - \alpha\beta + d\,(b+1) & -\alpha - d\beta + \alpha\,(b+1) \\ -ad - \alpha\beta + d\,(a+1) & -d^2 - \beta^2 + b\,(a+1) & -\beta - d\alpha + \beta\,(a+1) \\ -a\alpha - d\beta + \alpha\,(a+b) & -b\beta - d\alpha + \beta\,(a+b) & a + b - \alpha^2 - \beta^2 \end{bmatrix} = 0 \tag{A.10}$$

It is then easy to show that

$$a = \alpha^2 \tag{A.11a}$$

$$b = \beta^2 \tag{A.11b}$$

$$d = \alpha\beta \tag{A.11c}$$

matrix $M_0^{(1)}$ is then given by

$$M_0^{(1)} = \left(1 + \alpha^2 + \beta^2\right)^{-1} \begin{bmatrix} \alpha^2 & \alpha\beta & \alpha \\ \alpha\beta & \beta^2 & \beta \\ \alpha & \beta & 1 \end{bmatrix} \tag{A.12}$$

If the 33-term above dominates the matrix so that

$$\alpha < 1 \; and \; \beta < 1 \tag{A.13}$$

then we have a classic hierarchy pattern in $M_0^{(1)}$.

Appendix II

Equation (4.22) given by

$$-\frac{d\lambda_b}{dx} = \frac{1}{2}\lambda_b^3 + \frac{1}{12}\lambda_t^2\lambda_b \tag{B.1}$$

can be solved analytically by first substituting expression (2.3) for λ_t and solving

$$-\frac{dy}{dx} = \frac{1}{12}\lambda_t^2 y = \frac{1}{12}\lambda_{0t}^2\left(1+\lambda_{0t}^2 x\right)^{-1} y \tag{B.2}$$

which gives, with $C = constant$,

$$y = C\left(1+\lambda_{0t}^2 x\right)^{-\frac{1}{12}} \tag{B.3}$$

If we then take

$$\lambda_b(x) = \beta(x)\left(1+\lambda_{0t}^2 x\right)^{-\frac{1}{12}} \tag{B.4}$$

and normalize

$$\lambda_b(0) = \beta(0) = \lambda_{0b} \tag{B.5}$$

then the equation for $\beta(x)$ is obtained from (B.1) whose solution is then given by the following complicated relation

$$\frac{1}{\beta^2} = \frac{6}{5\lambda_{0t}^2}\left[\left(1+\lambda_{0t}^2 x\right)^{-\frac{5}{6}} - 1\right] + \frac{1}{\lambda_{0b}^2} \tag{B.6}$$

Substituting this value of $\beta(x)$ in equation (B.4) will then give us the complete analytic expression for λ_b

References

[1] P. Ramond, R. G. Roberts and G. G. Ross, Nucl. Phys. **B406**, 19 (1993)

[2] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983)

[3] M. H. Ahn *et al,* hep-ex / 0212007

[4] K. Eguchi *et al,* hep-ex / 0212021

[5] K. Scolderg , hep-ph / 9905016; T. Kajita, Invited talk at PASCOS 99, Y.Fukuda *et al,* Phys. Rev. Lett. **81**, 1562 (1998)

[6] Q. R. Ahmed *et al*, Phys. Rev. Lett. **87**, 071031 (2001); A. Bandyopadpadhyay, S. Choubey, S. Goswami, and D. P. Roy, Phys. Lett. **B540**, 14 (2002); V. Barger, D. Marfatia, K. Whisnat, and B. P. Wood, ibid. **B537**, 179 (2002); G. L. Fogli et al Phys. Rev **D66**, 053010 (2002); J. N. Bahcall, M. C. Gonzales-Garcia and C. Pena-Garay, hep-ph / 0204314

[7] M. Maltoni *et al,* hep-ph / 0212129; A. Bandyopadhyay *et al,* hep-ph / 0212146 ;J. N. Bahcall *et al* hep-ph / 0212147

[8] M. Appollonio *et al* Phys. Lett. **B466**, 415 (1999); F. Boehm *et al* Phys. Rev **D64**, 112001 (2001)

[9] See, for example, reviews by G. Altarelli and F.Ferugio, Phys. Rep. **320**, 295(1999) H. Fritzsch and Z. Xing, Pog. Part. Nucl. Phys. **45**,1 (2000), H. Murayama, Int. J. Mod. Phys. **A17**, 3403, (2002)

[10] B. Pontecorvo, JETP 34, **247** (1958); Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theo. Phys. **28**, 870 (1962)

[11] V. Barger, M. S. Berger and P. Ohlmann, Phys. Rev. **D47**, 1093 (1993)

[12] K.S. Babu, C. N. Leung and J. Pantaleone, Phys. Lett. **B319**,191(1993) P. Chankowski and Z. Pluciennick, Phys. Lett. **B316**, 312 (1993); S. Antusch, M. Drees, J.Kirsten, M. Lindner and M. Ratz, Phys. Lett. **B519**, 238 (2001);S. Antusch and M.Ratz, **JHEP0207**,059(2002)

[13] M. Gell-Mann, P. Ramond and R. Slansky, in *Supergarvity,* Proceedings of the Workshop, Stony Brook, New York 1979, edited by D. Z. Freeman and P. Van Nieuwenhuizen (North Holland, Amsterdam (1979); T. Yanagida, in Proceedings of the Workshop on the Unified Theory and Baryon Number in the Universe, edited by O. Sawada and A.Sugamoto (KEK, 1979); R.N. Mahopatra and G. Senjanovic, Phys. Rev. Lett. **44**,912 (1980)

[14] B. R. Desai and D. P. Roy, Phys. Rev. **D58**, 113007-1(1998)

[15] H. Fritzsch, Phys. Lett **B70**, 436 (1977) H. Fritzsch, Phys. Lett **B73**, 317(1978); H. Georgiand C. Jarlskog, *ibid.* **89B**, 297 (1979) ; P. Kaus and S. Meshkov, Mod. Phys.Lett. A3, 1251 (1988); E. Ma, Phys. Rev. **D43**, R2761(1991)

[16] T. K. Kuo, S. W. Mansour and G. H. Wu. Phys. Rev. **D60**, 093004 (1999); Phys. Lett. **B467**, 116 (1999); S. H. Chin, T. K. Kuo and G. H. Wu, Phys. Rev. **D62**, 053014 (2000)

[17] B. R. Desai and A. R. Vaucher, Phys. Rev **D63**, 13001-1 (2001)

[18] P.H. Frampton, S. L. Glashow, and D. Marfatia, Phys. Lett. **B536**, 79 (2002)

[19] B.R. Desai, D.P. Roy, and A.R. Vaucher, Mod. Phys. Lett. **A18** 1355(2003)

[20] W.A. Bardeen, C.T. Hill, and M. Lindner, Phys. Rev. **D41**, 1647 (1990)

[21] R. N. Mohapatra, M. K. Parida, and G. Rajasekaran hep-ph / 0301234 and references therein

[22] Particle Data Group K.Hagiwara *et al* Phys. Rev. **D66**, 010001 (2002)

[23] Y.Yamanaka, H. Sugawara, and S. Pakvasa, Phys. Rev. **D25**, 1895 (1982)

[24] E.Ma, Phys. Rev. **D43**, 2761 (1991); N. G. Deshpande, M. Gupta and P. B. Pal, ibid. **D45**, 953 (1992); K. S. Babu, E. Ma, and J.W.F. Valle, Phys. Lett. **B552**, 207 (2003)

[25] T. Brown, S. Pakvasa, H. Sugawara, and Y. Yamanaka, Phys. Rev. **D30**, 255 (1984)

[26] E. Ma, Phys. Rev. **D44**, 587 (1991), E. Ma, ibid **D61**, 033012-1(2000), E. Ma and G.Rajasekaran, ibid 113012-1 (2001)

[27] K. S. Babu and R. N. Mohapatra, Phys. Rev. Lett. **24**, 2418 (1995)

[28] P. H. Frampton and A.Rasin, Phys. Lett. **B478**, 424 (2000)

[29] B.R. Desai and G. Rajasekaran (work in progress)